Problem 1. Prove the following proposition.

Proposition. In an ordered field, the following properties hold:

(i) Unique identities. If \( a + x = a \) for every \( a \), then \( x = 0 \). If \( a \cdot x = a \) for every \( a \), then \( x = a^{-1} \).
(ii) Unique inverses. If \( a + x = 0 \), then \( x = -a \). If \( ax = 1 \), then \( x = a^{-1} \).
(iii) No divisors of zero. If \( xy = 0 \), then \( x = 0 \) or \( y = 0 \).
(iv) Cancellation for addition. If \( a + x = b + x \) then \( a = b \). If \( a + x \leq b + x \), then \( a \leq b \).
(v) Cancellation for multiplication. If \( ax = bx \) and \( x \neq 0 \), then \( a = b \). If \( ax \geq bx \) and \( x > 0 \), then \( a \geq b \).
(vi) \( 0 \cdot x = 0 \) for every \( x \).
(vii) \( -(−x) = x \) for every \( x \).
(viii) \( −x = (−1) \cdot x \) for every \( x \).
(ix) If \( x \neq 0 \), then \( x^{-1} \neq 0 \) and \( (x^{-1})^{-1} = x \).
(x) If \( x \neq 0 \) and \( y \neq 0 \), then \( xy \neq 0 \) and \( (xy)^{-1} = x^{-1}y^{-1} \).
(xi) If \( x \leq y \) and \( 0 \leq z \), then \( xz \leq yz \). If \( x \leq y \) and \( z \leq 0 \), then \( yz \leq xz \).
(xii) If \( x \leq 0 \) and \( y \leq 0 \), then \( xy \geq 0 \). If \( x \leq 0 \) and \( 0 \leq y \), then \( xy \leq 0 \).
(xiii) \( 0 < 1 \).
(xiv) For any \( x \), \( x^2 \geq 0 \).

Hint: You can prove these in any order you wish. Once you’ve proved a particular property, you can use it in your proof of a later property.

Proof. \( \square \) Suppose \( x + a = a \) for all \( a \), in particular \( a = 0 \). Then
\[
x = x + 0 = 0
\]
where the first equality follows from the identity property for 0. Similarly if \( xa = a \) for all \( a \), in particular \( a = 1 \), then
\[
x = 1x = 1.
\]
Suppose $a + x = 0$. Adding the additive inverse $(-a)$ of $a$ to both sides, we obtain

$$-a + (a + x) = -a + 0$$

$$\Rightarrow (-a + a) + x = -a$$

$$\Rightarrow 0 + x = -a$$

$$\Rightarrow x = -a,$$

using associativity and the identity property of 0 on the second line, the additive inverse property on the third, and the identity property on the fourth. Similarly if $ax = 1$ then

$$a^{-1}(ax) = a^{-1}1$$

$$\Rightarrow (a^{-1}a)x = a^{-1}$$

$$\Rightarrow 1x = a^{-1}$$

$$\Rightarrow x = a^{-1}$$

Since $0 + 0 = 0$ by the additive identity property, we have

$$0 \cdot x = (0 + 0) \cdot x = 0 \cdot x + 0 \cdot x.$$  

Adding the inverse $-(0 \cdot x)$ to both sides gives

$$-(0 \cdot x) + 0 \cdot x = -0 \cdot x + (0 \cdot x + 0 \cdot x)$$

$$\Rightarrow 0 = (-0 \cdot x + 0 \cdot x) + 0 \cdot x$$

$$\Rightarrow 0 = 0 \cdot x.$$  

Suppose $xy = 0$. If $x = 0$, we are done. If $x \neq 0$, then it has a unique multiplicative inverse $x^{-1}$. Multiplying by this element and using (vi), we have

$$y = 1 y = (x^{-1}x)y = x^{-1}(xy) = x^{-1}0 = 0.$$  

If $a + x \leq b + x$ then

$$(a + x) - x \leq (b + x) - x$$

$$\Rightarrow a + (x - x) \leq b + (x - x)$$

$$\Rightarrow a + 0 \leq b + 0$$

$$\Rightarrow a \leq b$$

and likewise with equality.

The additive inverse of $-x$ is the element $y = -(-x)$ such that

$$y + (-x) = 0.$$  

Since $y = x$ has this property, and additive inverses are unique by (ii), it follows that $x = -(-x).$
(viii) By distributivity and (vi),
\[ 0 = 0 \cdot x = (1 + (-1))x = 1x + (-1)x = x + (-1)x. \]
Since additive inverses are unique, it follows that \((-1)x = -x.\)

(ix) If \(x \neq 0\), then \(x^{-1}\) exists, and the multiplicative inverse of \(x^{-1}\) is the element \(y = (x^{-1})^{-1}\) such that
\[ yx^{-1} = 1. \]
Since \(y = x\) has this property and multiplicative inverses are unique by (iii), it follows that \(x = (x^{-1})^{-1}\). It must be true that \(x^{-1} \neq 0\) since otherwise we would have
\[ 1 = xx^{-1} = x0 = 0 \]
by (vi), which contradicts the axiom that \(0 \neq 1.\)

(x) Suppose \(x \neq 0\) and \(y \neq 0\). Then \(x^{-1}\) and \(y^{-1}\) exist, and by associativity and commutativity of multiplication,
\[ (x^{-1}y^{-1})(xy) = x^{-1}(y^{-1}x)y = x^{-1}(xy^{-1})y = (x^{-1}x)(y^{-1}y) = 1 \cdot 1 = 1. \]
By uniqueness of multiplicative inverses, it follows that \((xy)^{-1} = x^{-1}y^{-1}.\)

(xi) Suppose \(x \leq y\) and \(0 \leq z\). By compatibility of addition and order, we have
\[ 0 = x - x \leq y - x, \]
so \(0 \leq (y - x)\). By compatibility of multiplication (of positive elements) and order, and using distributivity, we have
\[ 0 \leq (y - x) \cdot z = yz + (-x)z. \]
By (viii), \((-x)z = ((-1)x)z = (-1)(xz) = -(xz).\) So now adding \(xz\) to both sides of the previous equation, using compatibility of addition and order, gives
\[ 0 + xz \leq (yz + (-x)z) + xz \]
\[ \implies xz \leq yz + (-xz) + xz \]
\[ \implies xz \leq yz. \]

(Lemma) Before proceeding, it will be convenient to use the fact that \((-1)(-1) = 1,\) or equivalently that the multiplicative inverse \((-1)^{-1}\) is \((-1).\) This follows from (viii) and (vii), namely \((-1)(-1) = -(-1).\)
(xii) Suppose \( x \leq 0 \) and \( y \leq 0 \). By adding the inverses to both sides of the inequalities, it follows that \( 0 \leq -x \) and \( 0 \leq -y \). Then by compatibility of the order and multiplication by positive elements,

\[
0 \leq (-x)(-y) \quad \text{by our Lemma.}
\]

If \( x \leq 0 \) and \( y \geq 0 \), then once again \( (-x) \geq 0 \), so

\[
0 \leq (-x)y = (−1) xy = −(xy).
\]

Adding \( xy \) to both sides, using compatibility of order and addition, we conclude

\[
xy \leq 0.
\]

(xiii) Proof by contradiction: Suppose \( 1 < 0 \). Then by (xii) \( 0 \leq 1 \cdot 1 = 1 \), which is a contradiction.

(v) First we claim that, in general, if \( a' \leq b' \) and \( c \geq 0 \) then \( a'c \leq b'c \). To prove this, note

\[
0 \leq (b' - a') \quad \Rightarrow \quad 0 = 0c \leq (b' - a')c \\
\quad \Rightarrow \quad 0 \leq b'c + (-a')c \\
\quad \Rightarrow \quad 0 \leq b'c + (−1)(a')c \\
\quad \Rightarrow \quad 0 \leq b'c - (a'c) \\
\quad \Rightarrow \quad a'c \leq b'c
\]

Now suppose \( ax \leq bx \) and \( x > 0 \). Since \( x \neq 0 \) there exists an inverse, and we claim that \( x^{-1} > 0 \) also. To see this, note that if \( x^{-1} \leq 0 \), then multiplying both sides by \( x \) and using what we just showed gives \( 1 = xx^{-1} \leq x \cdot 0 = 0 \), which contradicts (xiii).

Now it follows from what we showed above that

\[
(ax)x^{-1} \leq (bx)x^{-1} \\
\quad \Rightarrow \quad a \leq b.
\]

The case of equality is similar.

(xiv) This follows from (xii).

\( \square \)

**Problem 2.** Give an example of a field with only three elements. Prove that it cannot be made into an ordered field.
Solution. The field must have 0 and 1 as distinct elements, and then one other element we can call $a$. $\mathbb{F} = \{0, 1, a\}$ is then a field with respect to the following addition and multiplication tables:

\[
\begin{array}{c|ccc}
+ & 0 & 1 & a \\
\hline
0 & 0 & 1 & a \\
1 & 1 & a & 0 \\
a & a & 0 & 1 \\
\end{array}
\quad
\begin{array}{c|ccc}
\cdot & 0 & 1 & a \\
\hline
0 & 0 & 0 & 0 \\
1 & 0 & 1 & a \\
a & a & 0 & a \\
\end{array}
\]

In fact these are the only possible operations under which $\mathbb{F}$ is a field. It is reasonable to write $a = 2$ instead, and observe that the addition and multiplication here are given by the ordinary operations on integers, except that we always replace the result by the remainder that is left after dividing by 3. (This is called arithmetic ‘modulo 3’, and the integers modulo $p$ form a field whenever $p$ is a prime number.)

In any case, the order $0 < 1$ is forced on us by the field axioms, as proved above. So the only possibilities for the order are

(a) $a < 0 < 1$, in which case $1 = a + a < 0$ is a contradiction,
(b) $0 < a < 1$, in which case $1 = (a+a) < (1+a) = 0$ is a contradiction, or
(c) $0 < 1 < a$, in which case $a = (1+1) < (1+a) = 0$ is a contradiction. 

□

Problem 3. Show that $3^n/n!$ converges to 0.

Solution. Here is one solution. There exists an $N$ such that, for $n \geq N$, $n! \geq 6^n$. (To prove this, note that for $N = 14$, $N! > 6^N$. Then by induction if $n! > 6^n$ it follows that $(n+1)! > 6^{n+1}$ since $n+1 > 6$ for $n > 14$.) It follows that, for $n \geq N$,

$$0 \leq \frac{3^n}{n!} \leq \frac{3^n}{6^n} = \frac{1}{2^n}.$$  

since the sequences $x_n = 0$ and $y_n = 1/2^n$ both converge to 0, it follows by the Sandwich Lemma that $z_n = 3^n/n!$ converges to 0 as well. □

Problem 4. Let $x_n = \sqrt{n^2 + 1} - n$. Compute $\lim_{n \to \infty} x_n$.

Solution. First one has to determine what the limit will be. Since $\sqrt{n^2 + 1}$ behaves very much like $n$ as $n \to \infty$, we may guess that the limit will be 0. To actually prove this assertion, we must estimate the difference

$$\left| \sqrt{n^2 + 1} - n - 0 \right| = \sqrt{n^2 + 1} - n$$
as \( n \to \infty \). One trick to employ is the difference of squares formula \((a - b)(a + b) = a^2 - b^2\) where here \( a = \sqrt{n^2 + 1} \) and \( b = n \). Then

\[
\sqrt{n^2 + 1} - n = \frac{1}{\sqrt{n^2 + 1} + n}
\]

from which it is now more obvious that the sequence converges to 0, and which is more readily estimatable. For instance, since \( \sqrt{n^2 + 1} \geq 0 \), it follows that

\[
\sqrt{n^2 + 1} - n = \frac{1}{\sqrt{n^2 + 1} + n} \leq \frac{1}{n}.
\]

We could either appeal to the sandwich lemma, noting that \( 0 \leq x_n \leq \frac{1}{n} \) where the outermost sequences both converge to 0, or proceed directly.

For a direct proof, let \( \epsilon > 0 \) be given. Let \( N \) be an integer larger than \( 1/\epsilon \). Then

\[
\left| \left( \sqrt{n^2 + 1} - n \right) - 0 \right| = \sqrt{n^2 + 1} - n \leq \frac{1}{n} < \epsilon
\]

for \( n \geq N \). \( \square \)