Problem 1 (p. 97, #5). Let \( x_n \) be a monotone increasing sequence bounded above and consider the set \( S = \{x_1, x_2, \ldots\} \). Show that \( x_n \) converges to \( \sup(S) \). Make a similar statement for decreasing sequences.

Remark. This shows that the least upper bound property — that every nonempty set with an upper bound has a least upper bound — implies the monotone sequence property — that every monotone increasing bounded sequence bounded above converges. Combined with the reverse implication proved in class, it follows that the least upper bound property is equivalent to completeness.

Solution. \( S \) is a set with an upper bound, so it has a supremum \( x = \sup(S) \).

Let \( \varepsilon > 0 \). By our characterization of the supremum, there is some \( x_n \in S \) such that \( x - \varepsilon < x_n \) and since \( x_n \) is increasing it follows that

\[
  x - \varepsilon < x_n, \quad \forall \ n \geq N
\]

\[
  \implies |x_n - x| < \varepsilon, \quad \forall \ n \geq N
\]

Thus \( \lim_{n \to \infty} x_n = x = \sup(S) \).

If \( x_n \) is a decreasing sequence bounded below, then \( x_n \) converges to \( \inf(\{x_n\}) \) by a similar proof. \( \square \)

Problem 2 (p. 97, #7). For nonempty sets \( A, B \subset \mathbb{R} \), let \( A + B = \{x + y \mid x \in A \text{ and } y \in B\} \).

Show that \( \sup(A + B) = \sup(A) + \sup(B) \).

Solution. Let \( a = \sup(A) \) and \( b = \sup(B) \). Then since \( a \geq x \) for all \( x \in A \) and \( b \geq y \) for all \( y \in B \), it follows that \( a + b \) is an upper bound for \( A + B \), i.e.

\[
a + b \geq x + y, \quad \forall \ x + y \in A + B.
\]

Let \( \varepsilon > 0 \). Then there is some \( x \in A \) and \( y \in B \) such that \( a - \varepsilon/2 < x \) and \( b - \varepsilon/2 < y \), which means that

\[
a + b - \varepsilon < x + y \in A + B,
\]

and it follows that \( a + b = \sup(A + B) \). \( \square \)

Problem 3 (p. 52, #4).

(a) Let \( x_n \) be a Cauchy sequence. Suppose that for every \( \varepsilon > 0 \) there is some \( n > 1/\varepsilon \) such that \( |x_n| < \varepsilon \). Prove that \( x_n \to 0 \).

(b) Show that the hypothesis that \( x_n \) be Cauchy in (a) is necessary, by coming up with an example of a sequence \( x_n \) which does not converge, but which has the other property: that for every \( \varepsilon > 0 \) there exists some \( n > 1/\varepsilon \) such that \( |x_n| < \varepsilon \).
**Solution.** (a) Let $\varepsilon > 0$ be given. Since $x_n$ is Cauchy, there exists an $N$ such that $|x_n - x_m| < \varepsilon/2$ for all $m, n \geq N$. If we now let

$$\varepsilon_1 = \min(\varepsilon/2, 1/N)$$

then it follows from the other assumption that there is a $k > 1/\varepsilon_1 \geq N$ such that

$$|x_k| < \varepsilon_1 \leq \varepsilon/2.$$

Thus, for $x_n \geq N$, we have

$$|x_n - 0| = |x_n - x_k + x_k| \leq |x_n - x_k| + |x_k| < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

so that $x_n \to x$.

(b) Consider the sequence

$$x_n = \begin{cases} 1 & n \text{ odd} \\ 1/n & n \text{ even.} \end{cases}$$

This clearly does not converge, and yet for any $\varepsilon > 0$ we can choose an even $n > 1/\varepsilon$ for which $|x_n| < \varepsilon$.

\[\square\]

**Problem 4** (p. 99 #15). Let $x_n$ be a sequence in $\mathbb{R}$ such that $|x_n - x_{n+1}| \leq \frac{1}{2} |x_{n-1} - x_n|$. Show that $x_n$ is a Cauchy sequence.

**Solution.** To show that $x_n$ is Cauchy, we must compare $x_n$ and $x_m$ for all $n, m \geq N$ for various $N$, not just subsequent elements. To do this we first note that for arbitrary $k > 0$, let $M = |x_0 - x_1| \in \mathbb{R}$. Then for an arbitrary $\varepsilon > 0$, we may choose $N$ sufficiently large that $\frac{1}{2N-1} < \frac{\varepsilon}{M}$. (This uses the fact that $1/2^n \to 0$.) Thus for any $n, m \geq N$, supposing that $m \geq n$, we can write $m = n + k$ for some $k \geq 0$ and then

$$|x_n - x_m| = |x_n - x_{n+k}| = |x_{n+1} + \cdots + x_{n+k-1} + x_{n+k} - x_{n+k}|$$

$$\leq |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \cdots + |x_{n+k-1} - x_{n+k}|$$

$$\leq |x_n - x_{n+1}| + \frac{1}{2} |x_n - x_{n+1}| + \frac{1}{2} |x_{n+1} - x_{n+2}| + \cdots + \frac{1}{2^{k-1}} |x_{n+k-1} - x_{n+k}|$$

$$= (1 + \frac{1}{2} + \cdots + \frac{1}{2^{k-1}}) |x_n - x_{n+1}|$$

$$< 2 |x_n - x_{n+1}|$$

$$\leq \frac{2}{2^n} |x_0 - x_1| = \frac{1}{2^n-1} |x_0 - x_1|.$$

Let $M = |x_0 - x_1| \in \mathbb{R}$. Then for an arbitrary $\varepsilon > 0$, we may choose $N$ sufficiently large that $\frac{1}{2N-1} < \frac{\varepsilon}{M}$. (This uses the fact that $1/2^n \to 0$.) Thus for any $n, m \geq N$, supposing that $m \geq n$, we can write $m = n + k$ for some $k \geq 0$ and then

$$|x_n - x_m| = |x_n - x_{n+k}| < \frac{M}{2^{n-1}} \leq \frac{M}{2^{N-1}} < \varepsilon,$$

so $x_n$ is Cauchy.

\[\square\]

**Problem 5.** Prove that an Archimedean ordered field in which every Cauchy sequence converges is complete (i.e. has the monotone sequence property). Here are some suggested steps:

(a) Denote the field by $\mathbb{F}$, and suppose $x_n$ is a monotone increasing sequence bounded above by some $M \in \mathbb{F}$.
(b) Proceeding by contradiction, suppose $x_n$ is not Cauchy. Deduce the existence of a sub-sequence $y_k = x_{n_k}$ with the property that

$$y_k \geq y_{k-1} + \varepsilon, \quad \forall \ k$$

(1) for some fixed positive number $\varepsilon > 0$ which does not depend on $k$.

(c) Using the Archimedean property, argue that $y_k$ cannot be bounded above by $M$, hence obtaining a contradiction.

(d) Conclude that $x_n$ converges.

\textit{Proof.} Suppose $\mathbb{F}$ is Archimedean and has the property that every Cauchy sequence in $\mathbb{F}$ converges. Let $x_n$ be a monotone sequence in $\mathbb{F}$, with an upper bound $M$, and suppose that $x_n$ is not Cauchy. Then there exists an $\varepsilon > 0$ such that, for all $N \in \mathbb{N}$, there is a pair $n, m \geq N$ for which

$$|x_n - x_m| \geq \varepsilon.$$

(This is just the negation of the statement that $x_n$ is Cauchy.)

We construct a subsequence as suggested by the hint. Choose $n_1 = 1$ (really it doesn’t matter where you start), and by induction suppose that we have $n_1 < n_2 < \cdots < n_k$ such that $x_{n_k} \geq x_{n_k-1} + \varepsilon$. Set $N = n_k$; then by assumption there is a pair $n_{k+1}, m_{k+1} \geq n_k$ (and without loss of generality we can suppose that $n_{k+1} > m_{k+1}$) such that

$$|x_{n_{k+1}} - x_{m_{k+1}}| \geq \varepsilon;$$

$$\implies x_{n_{k+1}} \geq x_{m_{k+1}} + \varepsilon \geq x_{n_k} + \varepsilon$$

since the sequence is increasing. This completes the induction step and gives a subsequence $y_k = x_{n_k}$ satisfying (1), where $\varepsilon > 0$ is a fixed positive number, per our assumption that $x_n$ is not Cauchy.

Let $d = M - y_1$ be the distance from the first element of the subsequence to the upper bound for $x_n$. By the Archimedean property of $\mathbb{F}$, there exists some $N \in \mathbb{N}$ such that

$$N > d/\varepsilon, \quad \iff \varepsilon N > d.$$

By the property (1) on the subsequence $y_k$, it follows that

$$y_N \geq y_1 + N\varepsilon > y_1 + d = M,$$

Since $y_N = x_{n_N}$ is an element of the original sequence, this contradicts the assumption that $x_n$ is bounded.

Since we reached this conclusion by assuming that our bounded increasing sequence $x_n$ was not Cauchy, it follows that $x_n$ must be Cauchy, hence convergent by the assumption on $\mathbb{F}$. Since $x_n$ was an arbitrary increasing bounded sequence, it follows that $\mathbb{F}$ has the monotone sequence property. \[\square\]