Problem 1 (p. 172, #1). Which of the following sets are connected? Which are compact?

(a) \{ (x_1, x_2) \in \mathbb{R}^2 \mid |x_1| \leq 1 \}
(b) \{ x \in \mathbb{R}^n \mid \|x\| \leq 10 \}
(c) \{ x \in \mathbb{R}^n \mid 1 \leq \|x\| \leq 2 \}
(d) \mathbb{Z} = \{ \text{integers in } \mathbb{R} \}
(e) a \text{ finite set in } \mathbb{R}
(f) \{ x \in \mathbb{R}^n \mid \|x\| = 1 \} \text{ (Be careful with the case } n = 1! \)
(g) Boundary of the unit square in \mathbb{R}^2
(h) The boundary of a bounded set in \mathbb{R}
(i) The rationals in [0, 1]
(j) A closed set in [0, 1]

Solution.
(a) Connected, noncompact.
(b) Connected, compact.
(c) Compact. Connected if \( n \geq 2 \), not connected if \( n = 1 \).
(d) Not connected, not compact.
(e) Connected if just one point, otherwise not connected. Is compact.
(f) Compact. Connected if \( n \geq 2 \), not connected if \( n = 1 \), where the set is \{±1\} \subset \mathbb{R}.
(g) Connected, compact.
(h) Always compact (the boundary of a set \( A \) is always closed, being the intersection of closed sets \( \text{cl}(A) \) and \( \text{cl}(\mathbb{R} \setminus A) \), and is bounded if the set is bounded). May or may not be connected: for example \( \text{bd}([0, 1]) = \{0, 1\} \) is not connected, while \( \text{bd}(\{0\}) = \{0\} \) is connected.
(i) Neither connected nor compact.
(j) Compact; may or may not be connected.

\[ \square \]

Problem 2 (p. 191, #4). Let \( f : A \subset \mathbb{R}^n \rightarrow \mathbb{R} \) be continuous, \( x, y \in A \) and \( c : [0, 1] \rightarrow A \subset \mathbb{R}^n \) be a continuous curve joining \( x \) and \( y \). Show that along this curve, \( f \) attains its maximum and minimum values (among all values along the curve).

Solution. Since composition of continuous functions is continuous, \( f \circ c : [0, 1] \rightarrow \mathbb{R} \) is continuous. The domain \([0, 1]\) is compact, so \( f \circ c \) attains its maximum and minimum values (owing to compactness of \( f \circ c([0, 1]) \)). This is the same as the statement to be shown.

Problem 3 (p. 193, #3). Let \( f : [0, 1] \rightarrow [0, 1] \) be continuous. Prove that \( f \) has a fixed point (i.e. a point \( x \in [0, 1] \) such that \( f(x) = x \)).

Solution. Since \( f \) is continuous, \( g(x) = f(x) - x \) is continuous. A fixed point is the same thing as a point \( x_0 \in [0, 1] \) where \( g(x_0) = 0 \).

Suppose there are no fixed points. Since \( g([0, 1]) \) is connected, it must be that either \( g(x) > 0 \) for all \( x \in [0, 1] \) or \( g(x) < 0 \) for all \( x \in [0, 1] \). If \( g(x) > 0 \), then \( f(x) > x \) for all \( x \in [0, 1] \), but then \( f(1) > 1 \) which contradicts the assumption on the range of \( f \): that \( f : [0, 1] \rightarrow [0, 1] \). On the other hand, if \( g(x) < 0 \), then \( f(x) < x \) for all \( x \in [0, 1] \), but then \( f(0) < 0 \) which also contradicts the assumption on the range. Thus there must be some \( x \) such that \( g(x) = 0 \), or equivalently \( f(x) = x \).
Alternatively, we can note that \( g(0) = f(0) \in [0,1] \) and \( g(1) = f(1) - 1 \in [-1,0] \), and by the intermediate value theorem, for any \( c \in [g(1), g(0)] \), there exists \( x_0 \) such that \( g(x_0) = c \). In particular, \( c = 0 \) always lies in \([g(1), g(0)]\), so there exists a fixed point.

**Problem 4** (p. 174, #21).

(a) Prove that a set \( A \subset (M, d) \) is connected if and only if \( \emptyset \) and \( A \) are the only subsets of \( A \) that are open and closed relative to \( A \). (A set \( U \subset A \) is called open relative to \( A \) if \( U = V \cap A \) for some open set \( V \subset M \); ‘closed relative to \( A \)’ is defined similarly.)

(b) Prove that \( \emptyset \) and \( \mathbb{R}^n \) are the only subsets of \( \mathbb{R}^n \) that are both open and closed.

**Proof.**

(a) \( A \) is not connected if and only if there exist separating open sets \( U, V \subset M \) such that

1. \( A = (A \cap U) \cup (A \cap V) \),
2. \( A \cap U \neq \emptyset \),
3. \( A \cap V \neq \emptyset \),
4. \( (A \cap U) \cap (A \cap V) = \emptyset \).

Equivalently, \( U' = A \cap U \) and \( V' = A \cap V \) are nonempty, relatively open sets such that \( U' = A \setminus V' \) and \( V' = A \setminus U' \); in turn, this holds if and only if \( U' \) is a nonempty open set in \( A \) which is not all of \( A \) and which is both open and closed. Since all the implications are if and only if, the proof is complete.

(b) \( \mathbb{R}^n \) is path-connected, since any points \( x, y \in \mathbb{R}^n \) are connected by the path \( c(t) = (1 - t)x + ty \), hence is is connected. By part (a), it follows that the only subsets if it which are open and closed are \( \emptyset \) and \( \mathbb{R}^n \). 

**Problem 5.** Let \( (M_1, d_1) \) and \( (M_2, d_2) \) be metric spaces with compact sets \( K_1 \subset M_1 \) and \( K_2 \subset M_2 \).

Show that \( K_1 \times K_2 \) is a compact subset of the space \( (M_1 \times M_2, d = d_1 + d_2) \). (The metric \( d \) on the product \( M_1 \times M_2 \) is defined by \( d((x_1, x_2), (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2) \).)

**Solution.** By Bolzano-Weierstrass, we may replace ‘compact’ by ‘sequentially compact’. Let \((x_n, y_n)\) be a sequence in \( K_1 \times K_2 \). We are done if we show that it has a subsequence which converges in \( K_1 \times K_2 \).

Since \( K_1 \) is sequentially compact, there is a subsequence \( x_{n(k)} \) which converges in \( K_1 \):

\[ x_{n(k)} \xrightarrow{k \to \infty} x \in K_1. \]

Then consider the sequence \( y_{n(k)} \), \( k \in \mathbb{N} \), in \( K_2 \). Since \( K_2 \) is sequentially compact, this has a further subsequence \( y_{n(k(l))} \), \( l \in \mathbb{N} \) which converges in \( K_2 \):

\[ y_{n(k(l))} \xrightarrow{l \to \infty} y \in K_1. \]

The subsequence \( x_{n(k(l))} \) of \( x_{n(k)} \) also converges to \( x \) (since a subsequence of a convergent sequence always converges to the same limit), thus

\[ (x_{n(k(l))}, y_{n(k(l))}) \longrightarrow (x, y) \in K_1 \times K_2 \]

is a convergent subsequence of the original.